Numerical simulation of decaying axisymmetric turbulence

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Résumé

On étudie grâce à une méthode spectrale la turbulence axisymétrique en déclin, en géométrie cylindrique avec conditions aux limites périodiques dans la direction axiale. Les fonctions de base de notre étude sont des modes propres, orthonormaux, du rotationnel (fonctions de Chandrasekhar-Kendall). On observe la formation d'états auto-organisés quasi-stationnaires.

Abstract

A spectral-method is used to study decaying axisymmetric Navier-Stokes turbulence inside a cylindrical geometry with vertically periodic boundary condition. The trial functions are orthonormal eigenfunctions of the curl (Chandrasekhar-Kendall functions). The formation of quasi-stationary self-organized states is observed.

Key words :axisymmetric turbulence, spectral-method, Chandrasekhar-Kendall

1 Introduction

Whereas for 2D turbulence, statistical mechanics has been successfully applied to predict its self-organization [1, 2], such methods have not been able to successfully predict the dynamics of three-dimensional flows. Recently, progress has been made by studying the statistical mechanics of the axisymmetric Euler-Beltrami flows [3, 4, 5, 6], a case which is intermediate between 2D and 3D flows. The investigations have revealed characteristics of the thermodynamical equilibrium states considering a limited number of invariants (total Helicity H, total angular momentum I). In particular, it was shown that these quasi-stationary states (QSS) can be robust and significantly influence the dynamics of the flow.

So far, these results and predictions were obtained from statistical considerations, without taking into account the momentum equations. Whereas in a statistical sense axisymmetry is observed in a number of academical flows, instantaneously turbulent flows are never axisymmetric. In the present contribution we study the evolution of axi-symmetric turbulence by direct numerical simulation. Based on an existing numerical method applied to 2D turbulence [7, 8] and 3D magnetohydrodynamics [9, 10], we have modified and developed a spectral-method for axisymmetric Navier-Stokes turbulence, in a cylindrical geometry. The velocity field v is expanded as a Galerkin expansion in a set of orthonormal functions, in which the trial function is the Chandrasekhar-Kendall orthonormal eigenfunction of the curl [11]. Furtherly, the field is considered periodic in the vertical direction and has a 'no-penetration' boundary condition in the radial direction.

In Section 2, we first describe the computational method. As the method is entirely spectral, the Reynolds numbers are limited as compared to pseudospectral methods in a fully periodic case.

In Section 3, we show the results of the computation of axisymmetric flows at two different Reynolds numbers. As demonstrated by the former studies [3], the formation of coherent structures is observed, along with the approximate conservation of certain invariants associated with these QSS.

In Section 4, we summarize the results and propose perspectives.

2 Basics for the Computational Method

We begin from the Navier-Stokes equation in cylindrical coordinates (r,θ,z),

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{v} \times \boldsymbol{\omega} - \nabla p + \nu \nabla^2 \boldsymbol{v}, \tag{1}$$

and,

$$\nabla \cdot \boldsymbol{v} = 0. \tag{2}$$

In Eq.(1), $\boldsymbol{v}$ is the velocity field, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$ is the vorticity. p is the pressure divided by density, which can be obtained by taking the divergence of Eq.(1) and solving the Poisson equation which results when we consider the incompressible case. Nevertheless, as we will see afterward, axisymmetry will make p eliminable by projecting on a divergence-free orthonormal set of base functions, under a Galerkin approximation. Viscosity ν is taken to be dimensionless. The azimuthal derivatives $\partial_\theta = 0$

as we are considering an axisymmetric case.

The spectral technique which is implemented involves expanding $\boldsymbol{v}$ in terms of Chandrasekhar-Kendall functions of the curl [11]:

$$\boldsymbol{v}(r,z,t) = \sum_{nq} \xi_{nq}(t)\boldsymbol{A_{nq}}(r,z), \tag{3}$$

$$\boldsymbol{\omega}(r,z,t) = \sum_{nq} \lambda_{nq}\xi_{nq}(t)\boldsymbol{A_{nq}}(r,z), \tag{4}$$

Here, $\xi_{nq}(t)$ are scalar amplitudes(expansion coefficients) which depend on time.

The Chandrasekhar-Kendall functions: $\boldsymbol{A_{nq}}$ are defined by

$$\boldsymbol{A_{nq}} = I_{nq}^{-\frac{1}{2}}[\lambda_{nq}\nabla\psi_{nq} \times \boldsymbol{e_z} + \nabla \times (\nabla\psi_{nq} \times \boldsymbol{e_z})], \tag{5}$$

in a set of cylindrical orthonormal unit vectors ($\boldsymbol{e_r}$,$\boldsymbol{e_\theta}$,$\boldsymbol{e_z}$). The scalar function ψ_{nq} is a solution of the Helmholtz equation, $(\nabla^2 + \lambda^2)\psi = 0$. After the derivation, using that in our case the field is taken periodic in the vertical direction, and $\partial_\theta = 0$, the explicit form of ψ_{nq} turns out to be

$$\psi_{nq} = J_0(\gamma_{nq}r)e^{-ik_nz}. \tag{6}$$

Here J_0 is a Bessel function of the first kind of order 0. The wave number $k_n = 2\pi n/L_z$, where L_z is the length in the axial direction, $n = 1, \pm1, \pm2, \ldots$, $q = 1, 2, 3, \ldots$ corresponds to the values of λ_{nq}, $\lambda_{nq}^2 = \gamma_{nq}^2 + k_n^2$. As the velocity $\boldsymbol{v}$ is always a real number, and considering the conjugated expansion coefficients $\xi_{nq}(t)$, here λ_{nq} are determined to be always positive. I_{nq} is a normalization constant obtained by

$$\frac{1}{V}\int \boldsymbol{A_{nq}} \cdot \boldsymbol{A_{nq}^*} d^3x = 1, \tag{7}$$

where the volume integration is within the interior of the cylinder $0 \leq r < a$, $0 \leq z \leq L_z$, and $V = \pi a^2 L_z$.

That leaves the γ_{nq} to be specified. When $k_n \neq 0$, by the no penetration boundary conditions, $\boldsymbol{A_{nq}} \cdot \boldsymbol{e}_r(r = a) = 0$, it gives

$$J_1(\gamma_{nq}a) = 0, \tag{8}$$

which determines an infinite sequence of positive γ_{nq}, and the associated λ_{nq}.

When $k = 0$, there is no radial component, and some decisions are necessary. One convenient choice [9] is to demand the $\boldsymbol{A_{0q}}$ to be "flux-less" , which is achieved by using $\int \boldsymbol{e}_z \cdot \boldsymbol{A_{0q}} d^2x = 0$ to determine the γ_{0q} and λ_{0q}. After simple integration, it also leads to Eq.(8). Hence, $\gamma_{nq}a$ are taken as the zeros of Bessel function J_1. The normalization integral I_{nq} is

$$I_{nq} = 2\gamma_{nq}^2 \lambda_{nq}^2 J_0^2(\gamma_{nq}a). \tag{9}$$

All these choices allow us to show the orthonormality

$$\frac{1}{V}\int \boldsymbol{A_{n'q'}} \cdot \boldsymbol{A^*_{n''q''}} d^3x = \delta_{n'n''} \cdot \delta_{q'q''}. \tag{10}$$

Upon substitution of the expansion Eq.(3), Eq.(4) into Eq.(1), and then taking inner products one at a time with the individual A_{nq}, we find

$$\frac{\partial}{\partial t}\xi_{nq} = \frac{1}{V}\int_V \boldsymbol{A^*_{nq}} \cdot [(\boldsymbol{v} \times \boldsymbol{\omega}) - \nabla p + \nu \nabla^2 \boldsymbol{v}]. \tag{11}$$

Under a Galerkin approximation [12], we truncate the expansion and only retain a finite number of modes. As the result, we convert Eq.(1) into a set of ordinary differential equations for the expansion coefficient. Since $(\nabla p)_\theta = 0$ by axisymmetry, and the integration $\frac{1}{V}\int_V \boldsymbol{A^*_{nq}} \cdot (-\nabla p) = 0$, we find that in Eq.(11), p vanishes:

$$\begin{aligned} &\frac{\partial}{\partial t}\xi_{nq} + \nu\lambda_{nq}^2 \xi_{nq} = \\ &\sum_{n'q'}\sum_{n''q''} \lambda_{n''q''}\xi_{n'q'}\xi_{n''q''} \cdot \frac{1}{V}\int_V \boldsymbol{A^*_{nq}} \cdot (\boldsymbol{A_{n'q'}} \times \boldsymbol{A_{n''q''}}) d^3x. \end{aligned} \tag{12}$$

Eqs.(12) are solved numerically by a 4th-order Runge-Kutta method, with $a = L_z = 2\pi$. The convolution in 3D is directly evaluated since a fast algorithm like the fast-Fourier-transform does not exist for the Bessel functions. As a first check, the code is tested for the ideal case in which $\nu = 0$, and it is observed that the total energy is well conserved up to at least $\frac{dE}{dt} \leq 10^{-9}$ using dimensionless time units. The required resolution for our viscid computations was established by careful numerical convergence checks, increasing the resolution until all invariants evolved independently of the resolution.

3 Computational Results

We summarize the results of our simulations for $\nu = 0.01$ ($\mathrm{Re} \approx 80$ at $t = 0$) with $n_{max} = 10$ and $q_{max} = 40$, and $\nu = 0.001$ ($\mathrm{Re} \approx 800$ at $t = 0$) with $n_{max} = 25$ and $q_{max} = 60$. The same initial condition is used for these 2 runs. The initial conditions is generated by defining the energy of a small number of large-scale modes with a random phase. In Fig.(1) the initial $k_z - q$ spectrum is shown. The Reynolds number is here based on the radial integral length scale and the $\mathrm{r.m.s}$ of velocity.

The time histories of total energy E, helicity H and angular momentum I are shown in Fig.(2). Time is normalized by the integral timescale defined as $\tau = \frac{a}{u_{\mathrm{r.m.s}}}$. The initial value of E, H or I is the same for these 2 runs, since we used the same initial condition. A clear difference between the simulations is that at $\mathrm{Re} \approx 80$ all invariants decay rapidly to zero, whereas this is not the case for $\mathrm{Re} \approx 800$, where the quantities evolve to an approximately constant value. To explain the phenomenology behind this behavior, we show in Fig.(3) the contribution of the 4 modes wihich are dominating the angular momentum. It is observed that for the run at $\mathrm{Re} \approx 800$ an important increase of angular momentum is observed for mode $L(0, 1)$. This increase is due to nonlinear mode coupling. For longer times this mode dominates the dynamics since the viscous dissipation acts only weakly on this mode. Further verification of this 'selective-decay' scenario will be presented in a future paper. Fig.(4) and Fig.(5) show the evolution of the stream function and vorticity in the $r - z$ plane. A formation of large coherent-structures is observed, especially for the stream function. These observations are in agreement with the predictions of the statistical mechanics, already at the moderate Reynolds numbers used in the present investigation.

4 Conclusion and perspectives

In summary, a spectral-method based on Galerkin approximation is described and applied to calculate decaying axisymmetric Navier-Stokes turbulence. So far we have obtained results which seem to agree, at least qualitatively, with the results of statistical mechanics. In perspective, we will perform a more quantitative comparison with the theoretical studies and we will investigate the influence of the Reynolds number and of the invariants (E, I, H) values on the results by increasing the resolution and the Reynolds number. Most importantly, we hope to validate the predictions of theory [3], thereby assessing the validity of the assumptions to predict the evolution of viscous axisymmetric turbulence by statistical

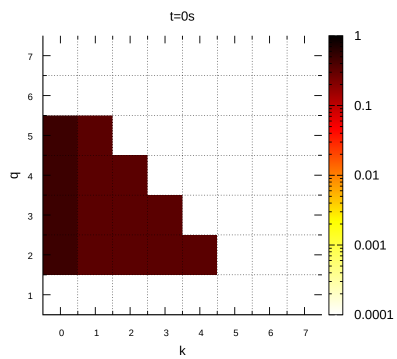

Figure 1: Initial $k_z - q$ spectral energy distribution.

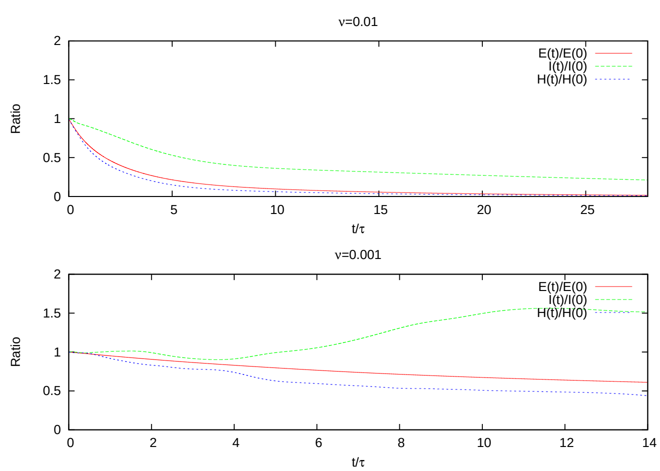

Figure 2: Time histories of the kinetic energy E, angular momentum I and Helicity H, normalized to their initial values. Top: $\nu = 0.01$; bottom: $\nu = 0.001$.

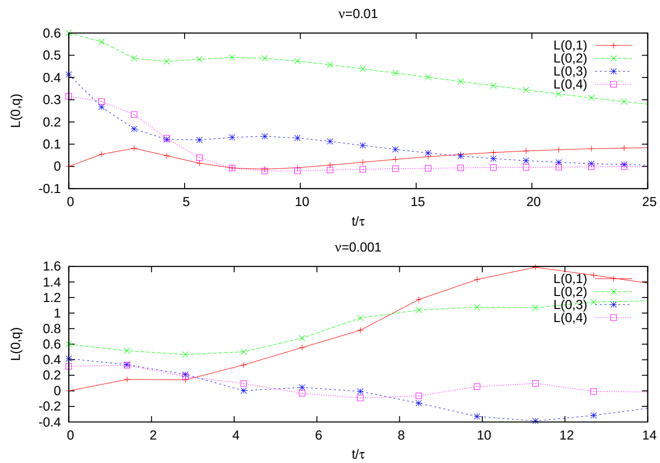

Figure 3: Modal angular momenta versus time normalized by the integral time. Top: $\nu = 0.01$; bottom: $\nu = 0.001$.

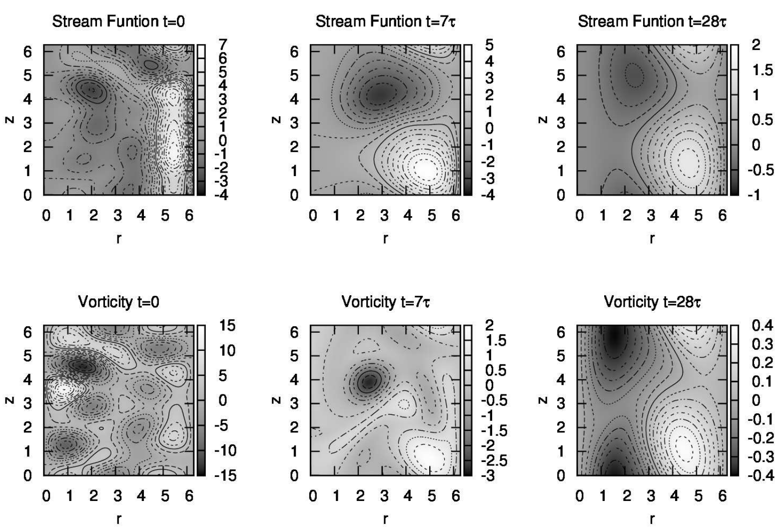

Figure 4: Evolution of Stream Function and Vorticity in the $r - z$ plane, $\nu = 0.01$.

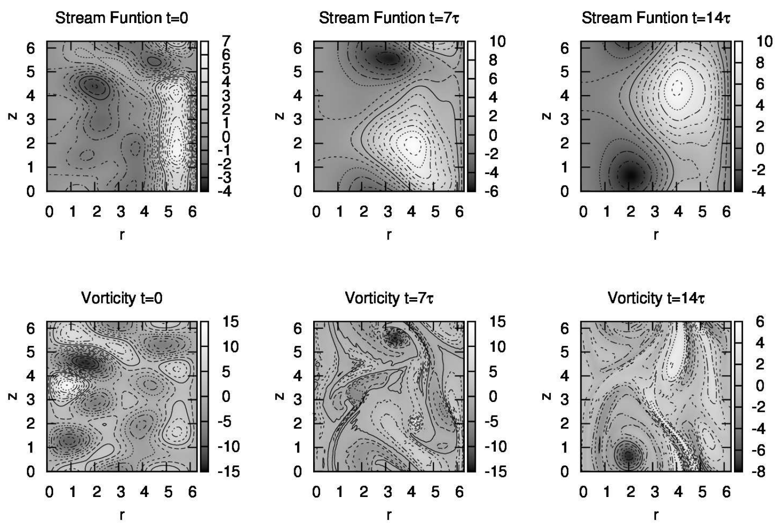

Figure 5: Evolution of Stream Function and Vorticity in the $r - z$ plane, $\nu = 0.001$.

mechanics.

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